D

[Carreux]

Caractérisat° de la loi normale. [Grimmett & Stirzaker - One thousand exercises in probability p. 262]

Théorème:

X, Y indépendantes, de même loi. On suppose $E(X) = 0$, $V(X) = 1$.
On suppose $X+Y \perp\!\!\!\perp X-Y$. Alors $X, Y \sim \mathcal{N}(0,1)$.

[G&S]

Dém: • $U = X+Y$, $V = X-Y$

$U \perp\!\!\!\perp V$ donc $\phi_{U+V} = \phi_U \phi_V$ càd $\phi_{2X} = \phi_{X+Y}\phi_{X-Y}$

$\phi_{2X} = \phi_X \phi_Y \phi_X \phi_{-Y}$.

$\phi(2t) = \phi(t)^3 \phi(-t) \quad \forall t \in \mathbb{R}$.

[Car]

• Mq $\phi(t) = \phi\left(\frac{t}{2^n}\right)^{a_n} \phi\left(\frac{-t}{2^n}\right)^{b_n}$, $a_n \geq 1, b_n \geq 1$, $\forall n \in \mathbb{N}^*$.

$\underline{n=1}$. $\phi(t) = \phi\left(\frac{t}{2}\right)^3 \phi\left(-\frac{t}{2}\right)$

$\underline{n \to n+1}$. $\phi(t) = \phi\left(\frac{t}{2^n}\right)^{a_n} \phi\left(\frac{-t}{2^n}\right)^{b_n}$

$$= \left[\phi\left(\frac{t}{2^{n+1}}\right)^3 \phi\left(\frac{-t}{2^{n+1}}\right)\right]^{a_n} \left[\phi\left(\frac{-t}{2^{n+1}}\right)^3 \phi\left(\frac{t}{2^{n+1}}\right)\right]^{b_n}$$

$$= \left[\phi\left(\frac{t}{2^{n+1}}\right)\right]^{\underbrace{3a_n+b_n}_{\geq 1}} \left[\phi\left(-\frac{t}{2^{n+1}}\right)\right]^{\underbrace{a_n+3b_n}_{\geq 1}}$$

Supposons $\phi(t) = 0$. $\forall n$, $\phi\left(\frac{t}{2^{n+1}}\right) = 0$ ou $\phi\left(\frac{-t}{2^{n+1}}\right) = 0$.
Or ϕ est C^0 en 0 et $\phi(0) = 1$. Absurde.

[G&S]

• On définit $\psi(t) = \dfrac{\phi(t)}{\phi(-t)}$.

Alors $\psi(2t) = \dfrac{\phi(2t)}{\phi(-2t)} = \dfrac{\phi(t)^3\phi(-t)}{\phi(-t)^3\phi(t)} = \dfrac{\phi(t)^2}{\phi(-t)^2} = \psi(t)^2$.

Donc $\psi(t) = \psi\left(\frac{t}{2}\right)^2 = \psi\left(\frac{t}{4}\right)^4 = \ldots = \psi\left(\frac{t}{2^n}\right)^{2^n}$. $\forall n \geq 0$.

Or $\phi(h) = \phi(0) + \phi'(0)h + \dfrac{\phi''(0)h^2}{2} + \underset{h \to 0}{o}(h^2)$

Voir post "indépendance et loi normale", les-maths.net, Mai 2010.

Or $\mathbb{E}(X) = -i\phi'(0)$ et $V(X) = -\phi''(0)$.

Donc $\phi(h) = 1 - \frac{1}{2}h^2 + \underset{h\to 0}{o}(h^2)$.

Donc $\psi(h) = \dfrac{\phi(h)}{\phi(-h)} = \dfrac{1 - \frac{1}{2}h^2 + o(h^2)}{1 - \frac{1}{2}h^2 + o(h^2)} = 1 + o(h^2)$

Donc $\psi(t) = \psi\left(\frac{t}{2^n}\right)^{2^n} = \left(1 + \underset{n\to\infty}{o}\left(\frac{t}{2^{2n}}\right)\right)^{2^n} = e^{2^n \ln\left(1 + o\left(\frac{t}{2^n}\right)\right)}$

Or $2^n \ln\left(1 + o\left(\frac{t}{2^n}\right)\right) = 2^n \ln\left(1 + \varepsilon\left(\frac{t}{2^n}\right)\frac{t}{2^n}\right) \sim t\,\varepsilon\left(\frac{t}{2^n}\right) \xrightarrow[n\to\infty]{} 0$

Donc $\psi(t) \xrightarrow[n\to+\infty]{} 1$ D'où $\psi(t) = 1 \ \forall t \in \mathbb{R}$.

Donc $\phi(t) = \phi(-t) \ \forall t \in \mathbb{R}$.

Donc $\phi(t) = \phi\left(\frac{1}{2}t\right)^3 \phi\left(-\frac{1}{2}t\right) = \phi\left(\frac{t}{2}\right)^4 = \phi\left(\frac{t}{4}\right)^{4^2} = \dots = \phi\left(\frac{t}{2^n}\right)^{4^n}$

Or $\phi\left(\frac{t}{2^n}\right) = 1 - \frac{1}{2}\left(\frac{t}{2^n}\right)^2 + o\left(\left(\frac{t}{2^n}\right)^2\right)$

$$\text{Donc } \phi(t) = \left[1 - \frac{t^2}{2.4^n} + o\left(\frac{t^2}{4^n}\right)\right]^{4^n} = e^{4^n \ln\left(1 - \frac{t^2}{2.4^n} + o\left(\frac{t^2}{4^n}\right)\right)}$$

$$\text{Or } 4^n \ln\left(1 - \frac{t^2}{2.4^n} + o\left(\frac{t^2}{4^n}\right)\right) \sim 4^n\left(\frac{-t^2}{2.4^n} + o\left(\frac{t^2}{4^n}\right)\right) \sim \frac{-t^2}{2} + o(t^2) \longrightarrow \frac{-t^2}{2}.$$

Donc $\phi(t) = e^{-\frac{t^2}{2}}$. $X, Y \sim \mathcal{N}(0,1)$ ▨

- Rényi: p 304. Suppose $E(X^2) < \infty$ auss.. Log illegal
- Ouvrard p. 280 Très long
- Corr: calculatoire, pb avec exponentielle

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